THE MEASUREMENT OF TIME

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Abstract. We present a definition of time measurement based on high energy photons and the fundamental length scale and show that, for macroscopic time, it is in accord with the Lorentz transformation of special relativity. To do this we define observer in a different way than in special relativity.

1. Introduction:

String theory and loop quantum gravity theory claim that space and time are ultimately discrete. In spite of this, however, there has not been a serious attempt to derive the continuum equations of General and Special Relativity from a discrete space perspective. The main objective of this note is to present a definition of time measurement based on the fundamental Planck length scale that, on macroscopic scales, is consistent with the Lorentz transformation that characterizes Special Relativity. To carry this out we define time as a property of space rather than as an independent coordinate. In our definition, time is measured at a spatial location of Planck length by the amount of energy of very high frequency photons it receives.

The notion of observer in Special Relativity requires synchronized clocks throughout space and an experimenter who has complete and immediate access to all the information of the synchronized clocks [4, p. 78]. In this note we present a different method of time observation.

2. Model for Time Measurement

We assume that space has a fundamental length scale, L_P, the Planck length. Furthermore, we assume that the fundamental length scale is observer independent as in doubly-special relativity, DSR, [1-3]. That is, any moving frame measures L_P as the smallest unit of length. If this were not the case then length contraction would contravene the notion of minimum length. With velocity limited by the speed of light, we have a minimum time unit, T_p. Time dilation can only increase T_p. Hence no assumption is necessary on a frame independent fundamental scale for time. We now state

Postulate 1: The Planck length is the smallest unit of length and this length is frame independent.

Einstein assumed that a photon is a bundle of energy localized in a small volume of space, and that it remains localized as it moves away from the source with velocity c. The Planck-Einstein equation relates the energy content E of the photon to its

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frequency ν by the equation $E = h\nu$. But what is meant by the frequency of a photon if it is localized in space? Since $\nu = c/\lambda$, where λ is the wavelength, a photon of frequency ν is one whose entire energy content is contained within a length $\lambda = c/\nu$, and the illusion of a wave is created by a stream of photons with the same frequency. From now on we let $\lambda = L_P$ and refer to such photons as photons of maximal energy or simply maximal photons. Then $\nu_P = c/L_P$ is the frequency corresponding to the smallest wavelength possible, and possessing the maximum energy possible: $E_P = hc/L_P = h/T_p = 7.6810^28eV$. Particles with very high energy (UHE) are known to exist [5-7].

In summary, we view a photon of wavelength L_P to mean that all of its energy is localized in L_P. It is convenient this think of this energy as the total area under a power form such as a normal density supported entirely inside a Planck length as depicted in Figure 1. The time it takes for this power form - moving at the speed of light - to pass through L_P is T_P. We summarize the foregoing in

Postulate 2: The maximum energy a photon is $E_P = hc/L_P$. At any location having Planck length, this energy allows the measurement of an amount of time equal to $T_P = h/E_P = L_P/c$.

We regard the time duration of an event as a property of the spatial location from where the event was observed. Let us consider the location $[0, L_P]$ in a stationary frame S as depicted in Figure 2. An event spanning N energy pulses is shown in Figure 2, resulting in NT_P units of time being measured at the location $[0, L_P]$. The energy graph shown in Figure 2 is the sum of the energy graphs of N maximal photons. Postulate 3: The time duration of an event measured at a location of

Planck length in any frame is the number of energy pulses E_P it receives multiplied by T_P.

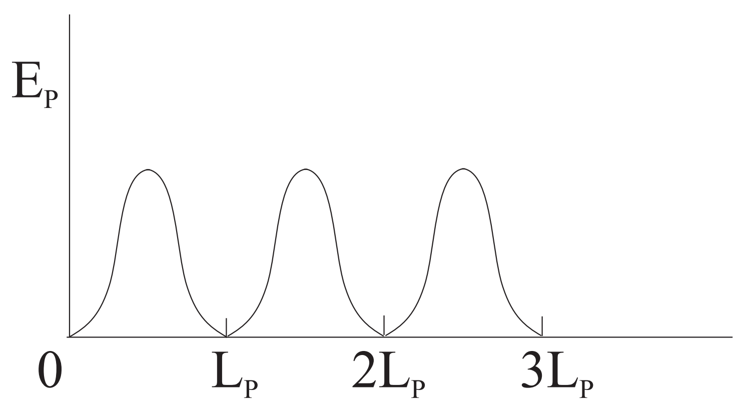

Figure 1: A maximal photon wave

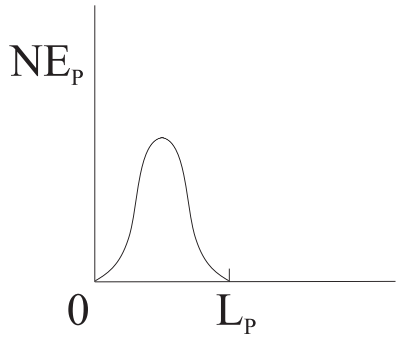

Figure 2: An event of duration NT_P at location $[0, L_P]$ interpreted as the reception of N pulses of energy from a stream of maximal photons.

3. Special Relativity

We now show that the foregoing definition of measuring time, when extended to macroscopic scales, satisfies the Lorentz transformation. Let us consider a frame S' moving with velocity v with respect to a stationary frame S. We consider a Planck length at an arbitrary location on the S' frame and at the same location on the S frame as is shown in Figure 3. In the S frame, an event on the S' frame is measured with relative velocity c-v. In the S' frame, it takes a maximal photon energy pulse $T_P^{'} = L_P^{'}/c$ units of time to pass through the Planck length. However, this same event, when measured from the same location on the S frame, measures the time of the energy pulse's passage as $L_P^{'}/(c-\mathrm{v})$. Since $L_P = L_P^{'}$ by Postulate 1, and $L_P^{'} = T_P^{'}\ c$, the time for the maximal photon to traverse L_P is $T_P^{'}\ c/(c-\mathrm{v})$. If we let $x = \mathrm{v}/c$, we obtain the relation between time observation in the S and S' frames at the same location of Planck length:

$$T_p = T_P^{'}/(1-x) \tag{1}$$

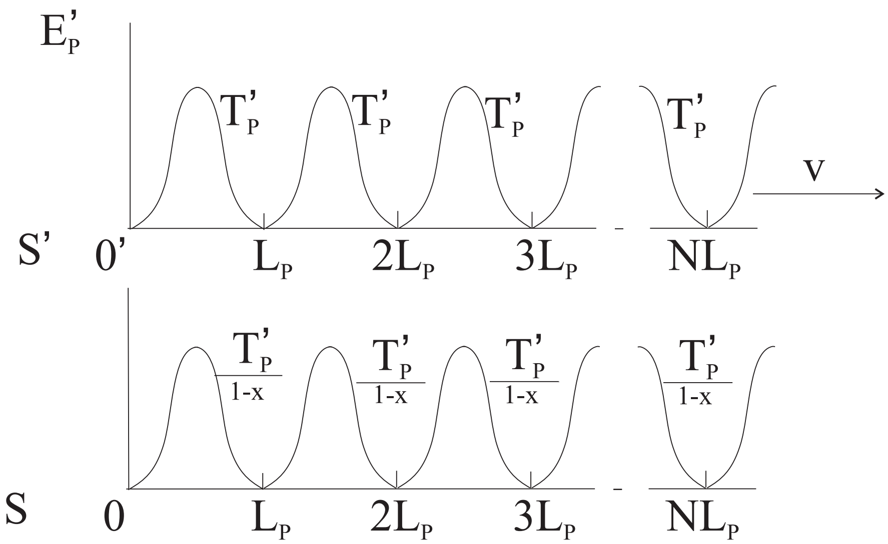

Figure 3: An event of N energy pulses as viewed in the Moving Frame S' and in the stationary frame S

We now imagine an event that spans N maximal photons as depicted in Figure 3. Let iL_P denote the ith Planck length in both frames when the two frames are aligned at their respective origins. Although the ith Planck length is the same in both frames the time duration of a maximal photon's passage through the ith Planck length is measured according to the reference frame. In the the S' frame it is $T_P^{'}$ while in S it is $T_p = T_P^{'}/(1-x)$. How does the observer, stationed at the origin of S, measure the time that is measured at the ith Planck length location in the S frame? It is reasonable to assume that the energy associated with $T_P^{'}$ is reduced and the degree of reduction depends on the location on the x-axis specified by i; the further away from the origin, the less energy will be received at the origin, and hence less time measured.

We model this process mathematically. Let $x = \mathrm{v}/c$ and note that the function $1/(1-x)$ in equation (1) is a continuous function on $(0,1)$. Let $C(0,1)$ denote the space of continuous functions on (0,1). Note the function $1/(1-x)$, which defines the time dilation due to one maximal photon and the relative motion between two frames at the same location. Our objective now is to define an operator A, which has the following properties: 1) For any function f in $C(0,1)$, with $f > 0$ and $f(0) = 1$, the iterate $A^i f$ denotes the amount of energy (time) transferred to the origin of S from the ith Planck length location in S. 2) The total measured time at the origin of S, for large scale times, should satisfy the Lorentz time dilation transformation. That is, if T' is the proper macroscopic time in the S' frame, then we want $T = T' \frac{1}{\sqrt{1-x^2}}$. This implies that $f^*(x) = \frac{1}{\sqrt{1-x^2}}$ is a fixed point of A. 3)We want f^* to be a stable fixed point of A. That is, f^* should be an attractor in $C(0,1)$ so that the approximations in using Planck scale lengths are valid.

Let $A : C(0,1) \to C(0,1)$ denote this operator and be defined by

$$Af(x) = \frac{1 + f^2(x) \cdot x^2}{f(x)}.$$

It is easy to very that f^* is a fixed point of A. It is our intention to prove that A satisfies all the desired properties and the iterates of A converge to f^*.

Proposition 1. *For any function $f(x) > 0$ with $f(0) = 1$ the iterations $A^n f$ converge to the function*

$$f^*(x) = \frac{1}{\sqrt{1-x^2}}.$$

The proof is a consequence of the following three lemmas. Let

$$A_x(a) = \frac{1 + a^2x^2}{a}\ , \quad 0 < x < 1\ , \quad a > 0\ .$$

Lemma 2. *a) If $a > \frac{1}{\sqrt{1-x^2}}$, then for those a satisfying this inequality we have $A_x(a) < a$ and $A_x^2(a) < a$.*

b) If $a < \frac{1}{\sqrt{1-x^2}}$, then for those a satisfying this inequality we have $A_x(a) > a$ and $A_x^2(a) > a$.

Proof. Let $a > \frac{1}{\sqrt{1-x^2}}$. Then, $a^2 > \frac{1}{1-x^2}$, or $a^2 - a^2x^2 > 1$, and $a > \frac{1+a^2x^2}{a}$. To prove the second statement we continue. We have $a^2x^2 > \left(\frac{1+a^2x^2}{a}\right)^2 x^2$, or $(1 + a^2x^2) \cdot \frac{a}{a} > 1 + \left(\frac{1+a^2x^2}{a}\right)^2 x^2$, which means $a > A_x^2(a)$.

The proof of the statement b) is similar. □

Lemma 3. *Let $g(x) = \frac{\sqrt{1-x^2}}{x^2}$. For $x < 1/\sqrt{2}$ we have $f^*(x) < g(x)$ and for $x > 1/\sqrt{2}$ we have $f^*(x) > g(x)$. For a between the graphs of f^* and g we have $A_x(a) \le f^*(x)$ and for the remaining a we have $A_x(a) > f^*(x)$.*

In particular: a) If $a < f^*(x)$ and $x < 1/\sqrt{2}$, then $A_x(a) > f^*(x)$.

b)If $a > f^*(x)$ and $x > 1/\sqrt{2}$, then $A_x(a) > f^*(x)$.

Proof. This follows by solving the inequality: $A_x(a) > \frac{1}{\sqrt{1-x^2}}$. □

Lemma 4. *If $a > 0$, then the sequence $\{A_x^n(a)\}_{n \ge 0}$, $0 < x < 1$, converges to $f^*(x) = \frac{1}{\sqrt{1-x^2}}$.*

Proof. First we consider $a > f^*(x)$.

Let us assume $x < 1/\sqrt{2}$. If $a > g(x)$ then the sequence $A_x^n(a)$ decreases until it goes to or below $g(x)$ at some step n_0. If $A_x^{n_0}(a) = g(x)$ then the next element $A_x^{n_0+1}(a) = f^*(x)$ and all the following elements have the same value. If $A_x^{n_0}(a) < g(x)$, then the following elements of the sequence oscillate below and above the value $f^*(x)$. By Lemma 2 a) the elements above $f^*(x)$ converge to this value monotonically. The "below" elements of the sequence also converge to the same limit since A_x is continuous.

For $x > 1/\sqrt{2}$ the sequence $A_x^n(a)$ is decreasing and converges to $f^*(x)$ monotonically.

Now, let $a < f^*(x)$. If $x < 1/\sqrt{2}$, then $A_x(a) > f^*(x)$ and we have convergence by the first part of the proof.

Let $x > 1/\sqrt{2}$. If $a \le g(x)$, then again $A_x(a) > f^*(x)$ and we have convergence by the first part of the proof. If $a > g(x)$, then the sequence $A_x^n(a)$ is increasing and converges to $f^*(x)$ monotonically. $\square$

The following theorem is the consequence of Proposition 1.

Theorem 5. *Let $f(x) = \frac{1}{1-x}$. We have the convergence of the averages*

$$\frac{1}{N}\left(f + A(f) + A^2(f) + \cdots + A^{N-1}(f)\right) \to f^*.$$

This means that

$$\frac{T_P^{'}}{N}[1/(1-x) + A^1(1/(1-x) + A^2(1/(1-x) + \cdots + A^{N-1}(1/(1-x)] \to \frac{T_P^{'}}{\sqrt{1-x^2}}$$

or

$$T_P^{'}[1/(1-x) + A^1(1/(1-x) + A^2(1/(1-x) + \cdots + A^{N-1}(1/(1-x)] \to \frac{NT_P^{'}}{\sqrt{1-x^2}}$$

that is, the total time observed at the origin of S approaches $\frac{NT_P^{'}}{\sqrt{1-x^2}}$ as the number of maximal photon pulses increases to ∞. But $NT_P^{'}$ is the proper time observed in the S' frame. Hence we have derived the Lorentz transformation for time dilation.

It is of interest to know how the measured dilation times approaches the Lorentz transformation. The following lemma shows that at all scales the dilation function stays above the graph of $\frac{1}{\sqrt{1-x^2}}$.

Proposition 6. *If $f \ge f^*$, then the sequence of averages stays above the limit function f^*:*

$$\frac{1}{N}\sum_{n=0}^{N-1} A^n(f) \ge f^* \ .$$

Proof. As we showed in Lemma 2, for $x > 1/\sqrt{2}$ and $a > f^*(x)$ the sequence $A_x^n(a)$ decreases monotonically and is above $f^*(x)$. The statement of the Lemma follows.

For $x < 1/\sqrt{2}$ and $a > f^*(x)$ the sequence $A_x^n(a)$ decreases monotonically for some time and then starts to oscillate below and above $f^*(x)$. We will prove that

$$a > f^*(x) \text{ and } A_x(a) < f^*(x) \quad \Longrightarrow \quad f^*(x) - A_x(a) < a - f^*(x) \ .$$

This implies the statement of the Lemma.

We want to show

$$\frac{1}{\sqrt{1-x^2}} - \frac{1+a^2x^2}{a} < a - \frac{1}{\sqrt{1-x^2}},$$

or

$$a^2(1+x^2) - a\frac{2}{\sqrt{1-x^2}} + 1 > 0.$$

Standard calculations show that this holds for $a > \frac{1}{\sqrt{1-x^2}}$. $\square$

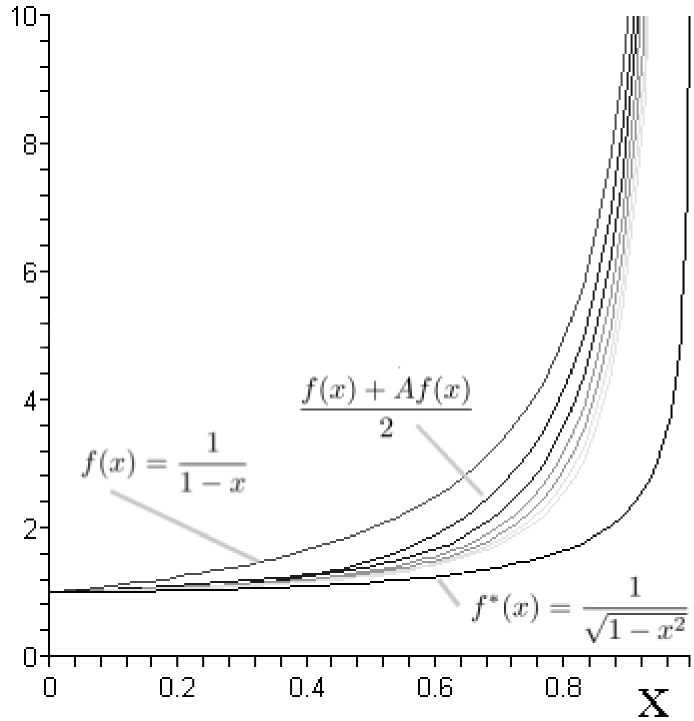

Figure 4: Averages $\frac{1}{N}\sum_{n=0}^{N-1} A^n(f)$, $N = 1, 2, \ldots, 7$, for $f(x) = \frac{1}{1-x}$(top curve) and the limit function $f^*(x) = \frac{1}{\sqrt{1-x^2}}$ (bottom curve).

Notes: 1.) Although in theory it is possible for maximal photons to exist, one may ask why they have not been observed. We suggest three possible answers: a)we do not have the experimental tools that can measure such small wavelengths, b) if the Planck-Einstein equation applies at the Planck length scale, then we do observe E_P in the form of time that we can measure, and c) on a more philosophical note, the energy E_P is the energy that allows a Planck length of space to be observed in time.

2.) Is time a dimension? The approach of this note argues against this. Time is not regarded as a coordinate. From our perspective, time is no different than color. Neither time nor color are independent variables; they are attributes of space. During an event, time accrues to Planck lengths in the form of energy. Thus, time is merely a number at every spatial location. When considered over many events, time at a Planck length location may be a fractal, due to the gaps between events and the various scales incurred by the duration of events.

3.) The cosmic ray paradox is concerned with why we are able to observe UHE particles. The tenor of this note is that such particles are ubiquitous and observable in the form of measurable time. But not all such particles adhere to Planck lengths of space and it is these particles that engender the cosmic ray paradox. We may ask why we cannot observe more of these UHE particles that are transformed into time. These particles create the observer's awareness of time and hence his consciousness.

This process is not measurable directly by the observer much as the high energy jolt of a defibrillator cannot be experienced by the person who is unconscious. It is only the much lower energies associated with living that can be experienced afterward.

4.) Dark energy may be energy carried by time much as ordinary energy is carried by mass.

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